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Note

On the asymptotic average length of a maximum common subsequence for words over a finite alphabet [1]

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Abstract

For a fixed subset A_n of the set C_n of words of length n over an alphabet with n symbols the following problem is considered: What is the asymptotic behaviour of the average length $AL(n)$ of a maximum common subsequence of words of C_n with those of A_n?

Complete answers are given for some nontrivial choices of A_n showing that $\lim_{n\to\infty} AL(n)$ may be (a) finite, but small; (b) finite, but arbitrarily large; (c) infinite.

1. Introduction

Let C_n denote the set of n^n words of length n over the alphabet $\mathbb{N}_n = \{1,2,\ldots,n\}$ and $A_n \subset C_n$ be a fixed subset. For every $c = c_1c_2\ldots c_n \in C_n$ we define

$$e(c) = \max\{1 \leqslant i \leqslant n\colon\ c_lc_{l+1}\ldots c_{l+i-1} = \varphi_j\varphi_{j+1}\ldots\varphi_{j+i-1},\ \text{for some } l,j \geqslant 1$$
$$\text{and } \varphi = \varphi_1\ldots\varphi_j\varphi_{j+1}\ldots\varphi_{j+i-1}\ldots\varphi_n \in A_n\}.$$

Let $c(n,k) = |\{c \in C_n\colon\ e(c) = k\}|$ and $d(n,k) = |\{c \in C_n\colon\ e(c) \geqslant k\}|$. It follows that $p(n,k) = c(n,k)n^{-n}$ is the probability that a word $c \in C_n$ has $e(c) = k$ and the average value $AL(n)$ of $e(c)$ is equal to

$$AL(n) = \sum_{k=1}^{n} kp(n,\, k). \tag{1}$$

In other words, $AL(n)$ is the average length of a maximum common subsequence of words of C_n with those of A_n. Because $c(n,k) = d(n,k) - d(n,k+1)$ for every

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$1 \leqslant k \leqslant n-1$, it follows that

$$AL(n) = n^{-n} \sum_{k=1}^{n} kc(n,k) = n^{-n} \sum_{k=1}^{n} d(n,k). \tag{2}$$

The problem we consider here is under what conditions on A_n there exists $\lim_{n\to\infty} AL(n)$ and in the affirmative case find this limit.

For every $1 \leqslant k \leqslant n$ we denote by L_k the set of pairwise distinct subwords of words of A_n consisting of k consecutive letters and let $s_k = |L_k|$. It is clear that $L_n = A_n$ and

$$c(n,n) = d(n,n) = |A_n| = s_n.$$

2. Computing methods for bounded $AL(n)$

First we shall indicate a direct way for computing $c(n,1)$. Let K_n^* be the complete digraph with vertex set $\mathbb{N}_n$ and edge set $E(K_n^*) = \{(i,j)\colon\ i,j \in \mathbb{N}_n\}$, having loops in every vertex. Now we define the digraph G as follows: $V(G) = \mathbb{N}_n$ and $E(G) = E(K_n^*) \setminus \{(i,j)\colon\ ij \in L_2\}$. Let A be the adjacency matrix of G.

Lemma 2.1. *We have*

$$c(n,1) = \sum_{i,j=1}^{n} (A^{n-1})_{ij} - (n-s_1)^n.$$

Proof. It is well known that $(A^{n-1})_{ij}$, the element in row i and column j in the matrix A^{n-1} is the number of walks of length $n-1$ from i to j in the graph G and any such walk corresponds (in a bijective way) to a word $c \in C_n$ such that $e(c) \leqslant 1$. In order to get the number of words c such that $e(c) = 1$ we must subtract the number of walks of length $n-1$ passing through no vertex of L_1, which is equal to $(n-s_1)^n$. $\square$

Hence $\lim_{n\to\infty} c(n,1)n^{-n}$ exists if and only if $\lim_{n\to\infty} n^{-n} \sum_{i,j=1}^{n} (A^{n-1})_{ij}$ exists and they are equal.

Lemma 2.2. *If $|L_1| = n$ and every letter in L_1 belongs to exactly r pairs in L_2, or equivalently, the digraph G associated with A_n is such that $d^+(x) = n-r$ for every $1 \leqslant r < n$, then $c(n,1) = n(n-r)^{n-1}$.*

Proof. We can compute $c(n,1)$ by Lemma 2.1 ($s_1 = n$), but a simpler counting argument is the following:

The first letter of a word $c \in C_n$ such that $e(c) = 1$ can be chosen in n ways, the second one in $n-r$ ways,..., the nth letter in $n-r$ ways, and the result follows. $\square$

A more general case is settled by the following lemma:

Lemma 2.3. *Suppose that there exist two fixed natural numbers, r, d, the sequence $(M_n)_{n\geqslant 1}$ of natural numbers such that $\lim_{n\to\infty} M_n = M < \infty$ and digraph G associated with A_n is such that $|\{x \in V(G)\colon\ d^+(x) \neq n-r\}| = M_n$ and $\forall x \in V(G), |d^+(x)-(n-r)| \leqslant d$. Then*

$$\lim_{n\to\infty} n^{-n}c(n,1) = \begin{cases} e^{-r} & \text{for } r>0, \\ 1-e^{-M} & \text{for } r=0. \end{cases}$$

Proof. Let first $r \geqslant 1$. We shall add and delete some directed edges from digraph G such that in the resulting digraph G_1 all vertices have their outdegrees equal to $n-r$. Added edges and deleted edges will be denoted by $e_1^+,\dots,e_p^+$ and $e_1^-,\dots,e_q^-$, respectively. Sequence $(M_n)_{n\geqslant 1}$ is bounded, i.e., there exists $K>0$ such that $M_n \leqslant K$ for every $n \geqslant 1$. It follows that $p+q \leqslant Kd$. By denoting $A_1^+,\dots,A_p^+$ and $A_1^-,\dots,A_q^-$ the sets of walks of length $n-1$ in G_1 and G, using directed edges $e_1^+,\dots,e_p^+$ and $e_1^-,\dots,e_q^-$, respectively, by Lemma 2.2 we find that

$$c(n,1) = n(n-r)^{n-1} - \left|\bigcup_{i=1}^{p} A_i^+\right| + \left|\bigcup_{j=1}^{q} A_j^-\right|.$$

But $|\bigcup_{i=1}^p A_i^+| \leqslant p \max_i |A_i^+| \leqslant p(n-1)n^{n-2}$, since two letters defining e_i^+ can be arranged on two consecutive positions in a word of length n in $n-1$ ways and the remaining $n-2$ letters can be chosen in at most n^{n-2} ways. In a similar way we get $|\bigcup_{j=1}^q A_j^-| \leqslant q(n-1)n^{n-2}$, hence in this case $\lim_{n\to\infty} c(n,1)n^{-n} = e^{-r}$. If $r=0$ it follows that $|L_1| = M_n$, hence the number of words $c \in C_n$ such that $e(c)=0$, i.e., they do not use any letter of L_1 is equal to $(n-M_n)^n$. It follows that

$$c(n,1) = n^n - (n-M_n)^n - \left|\bigcup_{i=1}^{p} A_i^+\right| + \left|\bigcup_{j=1}^{q} A_j^-\right|$$

and $\lim_{n\to\infty} c(n,1)n^{-n} = 1-e^{-M}$. □

Theorem 2.4. *If $|A_n| = \mathrm{O}(n^{1-\varepsilon})$ for some $\varepsilon>0$ or there exists $t \in \mathbb{N}, t \geqslant 2$ such that $\lim_{n\to\infty} s_i/n^{i-1} = 0$ for every $i=3,\dots,t+1$ and $|A_n| = \mathrm{O}(n^{t-\varepsilon})$ for some $\varepsilon>0$, then $\lim_{n\to\infty} AL(n)$ exists if and only if*

$$\lim_{n\to\infty} (d(n,1)+d(n,2))n^{-n}$$

exists and in the affirmative case they are equal.

Proof. For $k \geqslant 3$ let $L_k = \{w_1,\dots,w_{s_k}\}$, where each w_i is a word of length k. For every $1 \leqslant \alpha \leqslant n-k+1$ and $1 \leqslant p \leqslant s_k$ we denote

$$A^{(p)}_{\alpha,\alpha+1,\dots,\alpha+k-1} = \{c \in C_n\colon\ c_\alpha c_{\alpha+1}\dots c_{\alpha+k-1} = w_p\}.$$

It follows that

$$d(n,k) = \left|\bigcup_{p=1}^{s_k}\bigcup_{\alpha=1}^{n-k+1} A^{(p)}_{\alpha,\alpha+1,\dots,\alpha+k-1}\right| \leqslant s_k(n-k+1)n^{n-k} \tag{3}$$

since $|A^{(p)}_{\alpha,\alpha+1,\dots,\alpha+k-1}| = n^{n-k}$ for every $1 \leqslant \alpha \leqslant n-k+1$ and $1 \leqslant p \leqslant s_k$. Also we have

$$s_k \leqslant (n-k+1)|A_n| \tag{4}$$

since every $c \in A_n$ contains at most $n-k+1$ distinct subwords of length k. By denoting $f(n) = \sum_{i=3}^{n} d(n,i)n^{-n}$, we shall prove that $\lim_{n\to\infty} f(n) = 0$. If $|A_n| = O(n^{1-\varepsilon})$ then by (4):

$$\begin{aligned} f(n) &\leqslant s_3(n-2)n^{-3} + \sum_{k=4}^{n} s_k(n-k+1)n^{-k} \\ &\leqslant (n-2)^2|A_n|n^{-3} + n(n-3)^2|A_n|n^{-4} \\ &\leqslant 2|A_n|n^{-1} \to 0 \text{ as } n \to \infty. \end{aligned}$$

If there exists $t \in \mathbb{N}$, $t \geqslant 2$ such that $|A_n| = O(n^{t-\varepsilon})$ for some $\varepsilon > 0$ and $\lim_{n\to\infty} s_i n^{-(i-1)} = 0$ for every $3 \leqslant i \leqslant t+1$, we have

$$f(n) < \sum_{i=3}^{t+1} s_i n^{-(i-1)} + |A_n|n^{-t} + n|A_n|n^{-(t+1)}$$

and the proof follows. □

Note that if $s_1 = |L_1| = n$, then $d(n,1) = n^n$ and

$$d(n,1) + d(n,2) = 2d(n,1) - c(n,1) = 2n^n - c(n,1).$$

It follows that in this case if the conditions of Theorem 2.4 are satisfied and $\lim_{n\to\infty} AL(n)$ exists, then it is equal to $2 - \lim_{n\to\infty} c(n,1)n^{-n}$ and this limit can be computed by Lemma 2.3.

Corollary 2.5. *Suppose that at least one of the two conditions in Theorem* 2.4 *are satisfied. Then*

(a) *if* $\lim_{n\to\infty} s_1 = \sigma \in \mathbb{R}$ *then* $\lim_{n\to\infty} AL(n) = 1 - e^{-\sigma}$;

(b) *if the conditions in Lemma* 2.3 *relative to the outdegrees of vertices of G are satisfied and* $\lim_{n\to\infty} s_1 = \infty$, *then* $\lim_{n\to\infty} AL(n) = 2 - e^{-r}$.

Proof. (a) In this case the conditions in Lemma 2.3 are satisfied for $r = 0$, $M_n = s_1$ and $M = \sigma$, hence $\lim_{n\to\infty} c(n,1)n^{-n} = 1 - e^{-\sigma}$. We deduce $d(n,1) = n^n - (n-s_1)^n$, hence $\lim_{n\to\infty} d(n,1)n^{-n} = 1 - e^{-\sigma}$. Finally

$$\lim_{n\to\infty} AL(n) = \lim_{n\to\infty} 2d(n,1)n^{-n} - \lim_{n\to\infty} c(n,1)n^{-n} = 1 - e^{-\sigma}.$$

(b) Since every vertex x of G appearing in L_1 has a degree $d^+(x) \neq n$ it follows that $r > 0$. We deduce that $\lim_{n\to\infty} d(n,1)n^{-n} = 1$ since $\lim_{n\to\infty}(n-s_1)^n n^{-n} = 0$. Hence $\lim_{n\to\infty} AL(n) = 2 - e^{-r}$ by Lemma 2.3 and Theorem 2.4. □

Now we shall consider some examples.

Example 1. Let $A_n = \{11\ldots1, 22\ldots2, \ldots, nn\ldots n\}$. In this case $r = 1$, the complementary graph $\overline{G}$ consists of n loops, $|A_n| = n, s_3 = n$ and the conditions in Lemma 2.3 and Theorem 2.4 are satisfied for $r = 1$ and $t = 2$, respectively.

It follows that $\lim_{n\to\infty} AL(n) = 2 - e^{-1}$ by Corollary 2.5. Notice that in this case $e(c) = \max\{i\colon$ there exists j such that $c_j = c_{j+1} = \cdots = c_{j+i-1}\}$ and $2 - e^{-1}$ is also the asymptotic value of the average length of a maximum subword having equal letters of $c \in C_n$.

Example 2. Let $A_n = \{123\ldots n\}$. In this case $|A_n| = 1$, $\overline{G}$ consists of a path of length $n, r = 1, d = 1, M_n = 1$ for every n since in G we have $d^+(1) = \cdots = d^+(n-1) = n-1$ and $d^+(n) = n$. It follows that $\lim_{n\to\infty} AL(n) = 2 - e^{-1}$, hence the asymptotic average length of a maximum subword consisting of equal letters equals the asymptotic average length of a maximum subword consisting of consecutive letters.

Example 3. Let $A_n = \{11\ldots1\}$. In this case $\sigma = s_1 = 1$, hence $\lim_{n\to\infty} AL(n) = 1 - e^{-1}$.

Example 4. Let

$$A_n = \{\underbrace{11\ldots1}_{p}\underbrace{22\ldots2}_{q}\}$$

where $p, q \geqslant 2$ and $p + q = n$. In this case $|A_n| = 1$, $r = 0$, $s_1 = \sigma = 2$, $M = d = 2$ and $\lim_{n\to\infty} AL(n) = 1 - e^{-2}$.

Example 5. Let

$$A_n = \{\underbrace{1\ldots1}_{p_1}\underbrace{2\ldots2}_{q_1}, \underbrace{2\ldots2}_{p_2}\underbrace{3\ldots3}_{q_2}, \ldots, \underbrace{(n-1)\ldots(n-1)}_{p_{n-1}}\underbrace{n\ldots n}_{q_{n-1}}\},$$

where $p_i, q_i \geqslant 2$ and $p_i + q_i = n$ for every $1 \leqslant i \leqslant n - 1$. We have $|A_n| = n - 1, s_3 = O(n)$, $r = 2$, $M_n = M = 1$, $d = 1$. It follows that $\lim_{n\to\infty} AL(n) = 2 - e^{-2}$.

Lemma 2.6. *Let $d \in \mathbb{N}^*$ be a fixed number and*

$$A_n = \{\underbrace{a_1a_2\ldots a_{d-1}a_da_d\ldots a_d}_{n} : a_1, \ldots, a_d \in \mathbb{N}_n\}.$$

We have $\lim_{n\to\infty} AL(n) = d + 1 - e^{-1}$.

Proof. We have $s_{d+i} = n^d + O(n^{d-1})$ for every $i \geqslant 0$, hence $\lim_{n\to\infty} \sum_{k=d+2}^{n} d(n,k)n^{-n} = 0$ by (3). Also, $d(n,1) = d(n,2) = \cdots = d(n,d) = n^n$ and $d(n,d+1) = d(n,d) - c(n,d) = n^n - c(n,d)$. But $c(n,d) = n^d(n-1)^{n-d}$ since all words $c \in C_n$ having $e(c) = d$ may be obtained as follows: The first d

letters may be chosen in n^d ways and the letters on the $d+1, d+2, \dots, n$th position must be chosen such that each is different from the precedent one, hence in $n-1$ ways each.

It follows that $\lim_{n\to\infty} c(n,d) = e^{-1}$, which implies $\lim_{n\to\infty} AL(n) = d+1-e^{-1}$. $\square$

3. An example when $AL(n)$ is not bounded

If $\lim_{n\to\infty} |A_n| n^{-n} = \alpha > 0$ then $AL(n) \geqslant nc(n,n)n^{-n} = |A_n| n^{-(n-1)}$, hence $\lim_{n\to\infty} AL(n) = \infty$. We shall provide an example such that $\lim_{n\to\infty} |A_n| n^{-n} = 0$, but $\lim_{n\to\infty} AL(n) = \infty$.

A subword ij of $\sigma \in \mathbb{N}^*$ is called a level [2] if $i = j$ and a Smirnov word is a word with no levels. Let

$$A_n = \{c_1 c_2 \dots c_n\colon\ c_1 = c_2 \neq c_3 \neq \dots \neq c_n \text{ or } c_1 \neq c_2 \neq \dots \neq c_{n-1} = c_n,$$
$$c_i \in \mathbb{N}_n \text{ for } 1 \leqslant i \leqslant n\}. \tag{5}$$

It is clear that $|A_n| = 2n(n-1)^{n-2}$ for $n \geqslant 3$, hence $\lim_{n\to\infty} |A_n| n^{-n} = 0$.

Theorem 3.1. *For the choice* (5) *of* A_n *we have*

$$\lim_{n\to\infty} AL(n) = \infty.$$

Proof. We shall prove that for every fixed k, $k \geqslant 0$ we have

$$\lim_{n\to\infty} d(n,k+2) n^{-n} = 1. \tag{6}$$

Since $d(n,k+2) = n^n - \sum_{i=1}^{k+1} c(n,i)$, this is equivalent to

$$\lim_{n\to\infty} \sum_{i=1}^{k+1} c(n,i) n^{-n} = 0. \tag{7}$$

Since $c(n,1) = 0$ and $c(n,2) = n$ it follows that $d(n,2) = n^n - c(n,1) = n^n$ and $d(n,3) = n^n - n$, hence (6) can be verified directly for $k = 0$ and $k = 1$. Hence we shall consider further that $k \geqslant 2$. If c is a word of C_n such that $e(c) \leqslant k+1$, then c has the following structure:

$$\underbrace{SW_1}_{l_1} = \underbrace{\bullet = \cdots = \bullet}_{m_1} = \underbrace{SW_2}_{l_2} = \underbrace{\bullet = \cdots = \bullet}_{m_2} = \underbrace{SW_3}_{l_3} = \cdots$$
$$= \underbrace{SW_s}_{l_s} = \underbrace{\bullet = \cdots = \bullet}_{m_s}.$$

Here for $1 \leqslant i \leqslant s$, SW_i is a Smirnov word of length l_i; between these Smirnov blocks there are blocks of $m_1, \dots, m_s$ equal letters. Natural numbers $l_1, m_1, \dots, l_s, m_s$ must satisfy:

$$l_1 + m_1 + l_2 + m_2 + \cdots + l_s + m_s = n;$$

$$l_1 \geqslant 1;\ l_i \geqslant 2 \textit{ for every } 2 \leqslant i \leqslant s;\ l_j \leqslant k \textit{ for every } 1 \leqslant j \leqslant s \textit{ and}$$

$$m_i \geqslant 0 \textit{ for every } 1 \leqslant i \leqslant s. \tag{8}$$

If $l_1 = 1$, then the first $m_1 + 2$ letters of c are pairwise equal. This characterization holds because a word $c \in C_n$ has $e(c) \leqslant k+1$ relatively to choice (5) of A_n if and only if every maximal Smirnov block of c has length at most k. For a word of this type the ordered sequence $(l_1, m_1, \ldots, l_s, m_s)$ of length $2s$ will be called the pattern of c. It is clear that the number of words $c \in C_n$ having the pattern $(l_1, m_1, \ldots, l_s, m_s)$ is equal to

$$n(n-1)^{\sum_{i=1}^{s} l_i - s} \tag{9}$$

since the first letter of SW_1 can be chosen in n ways, the second one in $n-1$ ways, and so on, for SW_1 getting $n(n-1)^{l_1-1}$ ways; for any other block $SW_i (i \geqslant 2)$ we get $(n-1)^{l_i-1}$ possibilities of choosing its letters. From the conditions upon l_i and m_i we deduce that

$$1 \leqslant s \leqslant (n+1)/2. \tag{10}$$

Let us denote by $b(n,q)$ the number of words $c \in C_n$ such that $e(c) \leqslant k+1$, having in their patterns $\sum_{i=1}^{s} l_i = q$. It follows that

$$\sum_{i=1}^{k+1} c(n,i) = \sum_{q=1}^{n} b(n,q). \tag{11}$$

The number of compositions $l_1 + \cdots + l_s = q$ with parts $l_1, \ldots, l_s \geqslant 1$ is equal to $\binom{q-1}{s-1}$ and the number of compositions $m_1 + \cdots + m_s = n - q$ with parts $m_1, \ldots, m_s \geqslant 0$ is equal to $\binom{n-q+s-1}{s-1}$ (see e.g. [3]). Hence the number of patterns $(l_1, m_1, \ldots, l_s, m_s)$ satisfying the above-mentioned conditions is bounded above by

$$\binom{q-1}{s-1}\binom{n-q+s-1}{s-1}.$$

It follows that

$$b(n,q) \leqslant n(n-1)^{q-s}\binom{q-1}{s-1}\binom{n-q+s-1}{s-1}. \tag{12}$$

By denoting

$$F(q) = (n-1)^{q-s}\binom{q-1}{s-1}\binom{n-q+s-1}{s-1}$$

we shall prove that

$$\frac{F(q+1)}{F(q)} > 1 \text{ for every } 1 \leqslant q \leqslant n-1 \text{ and } n \geqslant 4. \tag{13}$$

Indeed,

$$\frac{F(q+1)}{F(q)} = (n-1)\frac{q}{q-s+1}\frac{n-q}{n-q+s-1} \geqslant (n-1)\frac{n-q}{n-q+s-1} = \frac{n-1}{1+\frac{s-1}{n-q}}$$

$$\geqslant \frac{n-1}{1+\frac{\frac{n+1}{2}-1}{n-q}} = \frac{n-1}{1+\frac{n-1}{2(n-q)}} \geqslant \frac{n-1}{1+\frac{n-1}{2}} = \frac{2(n-1)}{n+1} > 1$$

for $n > 3$ by (10), since $n - q \geqslant 1$ and (13) is proved.

Now we shall consider three cases: (A) $q = n$; (B) $q = n-1$ and (C) $q \leqslant n-2$.

(A) If $q = n$ from (12) it follows that

$$b(n,n) \leqslant n(n-1)^{n-s}\binom{n-1}{s-1} \leqslant \frac{n(n-1)^{n-1}}{(s-1)!} \tag{14}$$

since

$$\binom{n-1}{s-1} \leqslant \frac{(n-1)^{s-1}}{(s-1)!}.$$

In this case we have also $n = \sum_{i=1}^{s} l_i \leqslant ks$, hence $s \geqslant n/k$, which implies (by (14))

$$\lim_{n\to\infty} b(n,n)n^{-n} = 0.$$

(B) In this case

$$b(n,n-1) \leqslant sn(n-1)^{n-1-s}\binom{n-2}{s-1} \leqslant 2n(n-1)^{n-3}(n-2)$$

for every $s \geqslant 1$ and $n \geqslant 4$, hence $\lim_{n\to\infty} b(n,n-1)n^{-n} = 0$.

(C) If $q \leqslant n-2$ by (13) one deduces that

$$b(n,q) \leqslant n(n-1)^{n-s-2}\binom{n-3}{s-1}\binom{s+1}{2} \leqslant 3n(n-3)(n-1)^{n-4}$$

for every $s \geqslant 1$ and $n \geqslant 5$. Finally, $\sum_{i=1}^{k+1} c(n,i)n^{-n} = \sum_{q=1}^{n} b(n,q)n^{-n} \leqslant b(n,n)n^{-n} + b(n,n-1)n^{-n} + 3n^2(n-3)(n-1)^{n-4}n^{-n}$ for every $n \geqslant 5$, from which (7) follows. □

Problem 1. When A_n is given by (5) see whether $\lim_{n\to\infty} AL(n)n^{-1}$ exists. In the affirmative case find its value.

Problem 2. Determine the asymptotic behaviour of $AL(n)$ in the case of the symmetric group S_n.

Acknowledgements

The problem discussed in this paper has its origins into a question raised by Doran [1] about the asymptotic behaviour of the average length of a maximum subsequence consisting of equal letters of words of length n over an alphabet with n letters as $n \to \infty$. This question has some relevance in designing an efficient computing arhitecture for parallel computation. The author wishes to thank R.W. Doran and also C. Calude for setting the problem into a more general context.

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